# Calculus 2

Math 156 WVU

## 1 Introduction

The integral formalizes the intuitive concept of area. It turns out that defining such an intuitive notion is not an easy task. We will only be interested in defining the area of some special regions in the plane, namely those which are bounded by the graph of a sufficiently nice function $f$, the $x$-axis and the vertical lines $x = a$ and $x = b$. This notion of area comes with the understanding that areas below the $x$-axis are assigned a negative value. Let's review the idea behind the integral.

Starting with a function $f$ on $[a, b]$, we partition the domain into finitely many small subintervals. On each subinterval $[x_{i-1}, x_i]$, we pick some point $x_i^* \in [x_{i-1}, x_i]$ and use the value $f(x_i^*)$ as an approximation for $f$ on $[x_{i-1}, x_i]$. In other words, we build a row of thin rectangles to approximate the area between $f$ and the $x$-axis. The area of each rectangle is $f(x_i^*)(x_i - x_{i-1})$, and so the total area of all rectangles is given by

$$S = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

It is reasonable to expect that the accuracy of the approximation provided by $S$ improves as the rectangles get thinner. This suggests that we should take some sort of "limit" of these approximating sums as the width of the subintervals tends to zero. If it exists, this "limit" is the definition of the area under $f$.

We first have to precisely define such a limit. We then have to decide for which functions such a limit exists. Before embarking on these tasks, let's see a concrete example.

**Example 1.** Suppose we want to give a meaning to the area of the region between the graph of the function $f(x) = e^x$, the $x$-axis, the $y$-axis and the line $x = a$. Let's take the very simple subdivision of $[0, a]$ into $n$ subintervals of equal length. Clearly, the endpoints of such subintervals are at $0, a/n, 2a/n, \dots, a$. For each of them we construct a rectangle having as base the interval $[ka/n, (k+1)a/n]$ and whose height is the value of the function at the left endpoint, namely $e^{ka/n}$. The sum of the areas of these rectangles is given by

$$R_n = \sum_{k=0}^{n-1} \frac{a}{n} e^{ka/n}.$$

This sum is quite messy and since in the end we would like to study the value of $R_n$ for $n \to \infty$, i.e. when the intervals get thinner and thinner, we should figure out if it can be rewritten in a nice way. Notice that we are essentially interested in evaluating a sum of the form $1 + r + r^2 + \cdots + r^{i-1} + r^i$, usually called a geometric sum. We would then substitute the value $e^{a/n}$ for $r$ and $n - 1$ for $i$. The little trick is to multiply $1 + r + r^2 + \cdots + r^{i-1} + r^i$ by $r - 1$. Indeed, we obtain $(1 + r + r^2 + \cdots + r^{i-1} + r^i)(r - 1) = r^{i+1} - 1$ and so, if $r \neq 1$, we have

$$1 + r + r^2 + \cdots + r^{i-1} + r^i = \frac{r^{i+1} - 1}{r - 1}.$$

Coming back to our problem, we can then write

$$R_n = \sum_{k=0}^{n-1} \frac{a}{n} e^{ka/n} = \frac{a}{n} \cdot \frac{e^a - 1}{e^{a/n} - 1} = (e^a - 1) \cdot \frac{1}{\frac{e^{a/n} - 1}{\frac{a}{n}}}.$$

Let's now try to answer our initial question: What happens to $R_n$ for $n \to \infty$? Notice that for $n \to \infty$, we have that $a/n \to 0$. Therefore, what we really care about is

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

By a simple application of L'Hospital's rule, the value of this limit is $1$ and so

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(e^a - 1)\cdot \frac{1}{\frac{e^{a/n}-1}{\frac{a}{n}}} = e^a - 1.$$

Recall that this means that for every $\varepsilon > 0$, there exists a number $N_\varepsilon$ such that for all $n > N_\varepsilon$, we have

$$|R_n - (e^a - 1)| < \varepsilon.$$

In words, the values of $R_n$ can be made arbitrarily close to $e^a - 1$ by choosing $n$ sufficiently large.

Let's now consider rectangles having again as bases the subintervals $[ka/n, (k+1)a/n]$ but whose heights are the values of the function at the right endpoints, namely $e^{(k+1)a/n}$. In this case, the sum of the areas of the rectangles is given by

$$R_n' = \sum_{k=0}^{n-1} \frac{a}{n} e^{(k+1)a/n} = \sum_{k=0}^{n-1} \frac{a}{n} e^{ka/n} \cdot e^{a/n} = e^{a/n} \sum_{k=0}^{n-1} \frac{a}{n} e^{ka/n} = e^{a/n} \cdot R_n.$$

As $n \to \infty$, we have that $R_n'$ tends to $e^a - 1$ as well.

Coincidence? Is this true for all other possible partitions and choices of the sample points? It will turn out that the answers to these questions are no and yes, respectively.

Let us formalize the reasonings above in the following very long definition:

**Definition 1.** If $I = [a, b]$ is a closed bounded interval of real numbers, then a **partition** of $I$ is a $n+1$-tuple $\mathscr{P} = (x_0, x_1, \ldots, x_n)$ of points in $I$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

The $x_i$'s divide the interval $I$ into $n$ subintervals $I_1 = [x_0, x_1], \ldots, I_n = [x_{n-1}, x_n]$. The **norm** of $\mathscr{P}$ is the number

$$\|\mathscr{P}\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\},$$

i.e. the length of the largest subinterval into which the partition divides $[a, b]$.

An arbitrary point $x_i^*$ in the subinterval $I_i = [x_{i-1}, x_i]$ is called a **tag** (or **sample point**) of $I_i$. A partition $\mathscr{P}$ of $I$ together with a set of tags $\{x_1^*, \ldots, x_n^*\}$ such that $x_i^*$ belongs to $I_i$, for each $i = 1, \ldots, n$, is called a **tagged partition** of $I$ and denoted by $\dot{\mathscr{P}}$. Note that a tag could be any point in the subinterval and so an endpoint of a subinterval could be used as a tag for two consecutive subintervals. The norm of the tagged partition $\dot{\mathscr{P}}$ is defined as the norm of $\mathscr{P}$, i.e. it does not depend on the choice of tags.

The **Riemann sum** of a function $f\colon [a, b] \to \mathbb{R}$[1] corresponding to the tagged partition $\dot{\mathscr{P}}$ is the number

$$S(f, \dot{\mathscr{P}}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

As in our process of defining the area under the graph of a function we are working with limits, we should expect that not all quantities involved exist. Loosely speaking, we say that a function $f$ is Riemann integrable if the Riemann sum can be made as close as desired to a given quantity, called the integral of $f$ or the area under the graph of $f$, by making the subintervals small enough. This is formalized as follows:

**Definition 2.** A function $f\colon [a, b] \to \mathbb{R}$ is **Riemann integrable** on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$|S(f, \dot{\mathcal{P}}) - L| < \varepsilon. \tag{1}$$

The value $L$ is called the **Riemann integral** of $f$ over $[a, b]$. Instead of $L$, we will usually write

$$\int_a^b f(x)\,\mathrm{d}x \quad \text{or} \quad \int_a^b f.$$

[1] In words, a function defined on $[a, b]$ and having real values.

What did we verify in Example 1? Well, we showed that for every $\varepsilon > 0$, there exists a number $a/N_\varepsilon$ such that for every tagged partition with subintervals of the same length, all tags in the left or right endpoints and norm smaller than $a/N_\varepsilon$, we have that the corresponding Riemann sum $R_n$ satisfies

$$|R_n - (e^a - 1)| < \varepsilon.$$

So we proved less than what we really need in order to show that the area of the region between the graph of $f(x) = e^x$, the $x$-axis, the $y$-axis and the line $x = a$ is $e^a - 1$, or equivalently that

$$\int_0^a e^x = e^a - 1.$$

**Remark 3.** You might imagine that there is a game going on in Definition 2, played by an Engineer and a Hater. The Engineer is attending all Calculus 2 classes and he/she is sure that $\int_a^b f = L$. The Hater of course wants to challenge the Engineer and so they decide to play the following game:

- The Hater chooses an $\varepsilon > 0$.
- The Enginneer wisely looks at $\varepsilon$ and chooses a $\delta_\varepsilon > 0$.
- The Hater chooses a tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ with norm less than $\delta_\varepsilon$.

If $|S(f, \dot{\mathcal{P}}) - L| < \varepsilon$, the Engineer wins and the Hater looses, otherwise the Hater wins and the Engineer looses. Definition 2 tells us that if the Engineer plays cleverly, then he/she always wins. In other words, the Engineer has a winning strategy and the strategy consists in just picking an appropriate value of $\delta_\varepsilon$ for which (1) holds.

**Remark 4.** It is crucial for the success of a theory of integration that we require that every tagged partition of norm smaller than $\delta_\varepsilon$ satisfies (1). We would not get a satisfactory theory by only requiring that for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ and some tagged partition with norm less than $\delta_\varepsilon$ satisfying (1).

In principle, there might exist more than one $L$ satisfying (1). It can be verified that, if it exists, the integral is unique:

**Lemma 2.** *If $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then the value of the integral is uniquely determined.*

Using the definition of Riemann integrability it is not difficult to show that the following properties hold. Note that all of them are clearly satisfied by our intuitive notion of area.

**Theorem 3.** *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be two Riemann integrable functions on $[a, b]$. The following hold:*

1. *If $k \in \mathbb{R}$, the function $kf$ is Riemann integrable on $[a, b]$ and*

$$\int_a^b kf = k \int_a^b f.$$

2. *The function $f + g$ is Riemann integrable on $[a, b]$ and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

3. *If $f(x) \le g(x)$ for each $x \in [a, b]$, then*

$$\int_a^b f \le \int_a^b g.$$

**Example 4.** The constant function $f(x) = c$ is Riemann integrable on $[a, b]$ and $\int_a^b f(x)\,\mathrm{d}x = c(b - a)$. Indeed, for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$, we have

$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b - a).$$

Therefore, by taking $L = c(b - a)$, we have that (1) translates into $0 < \varepsilon$, which is true by assumption.

We have just seen that constant functions are Riemann integrable. If you feel particularly bold you might ask: Are all functions Riemann integrable?

**Example 5.** The following function $f\colon [0, 1] \to \mathbb{R}$ (called Dirichlet's function) is not Riemann integrable:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \text{ is a rational number;} \\ 0 & \text{if } x \in [0,1] \text{ is an irrational number, i.e. a number which is not a rational number.} \end{cases}$$

The idea is to suppose that $f$ is Riemann integrable and see how this leads to a nonsense. This would then imply that $f$ cannot be Riemann integrable.

Therefore, from now on we suppose that $f$ is Riemann integrable and let $L$ be the Riemann integral of $f$ over $[0, 1]$. Observe first that if $\dot{\mathcal{P}}$ is any tagged partition of $[0, 1]$ all of whose tags are rational numbers, then

$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = 1.$$

On the other hand, if $\dot{\mathcal{Q}}$ is any tagged partition all of whose tags are irrational numbers, then $S(f, \dot{\mathcal{Q}}) = 0$. We now need some hand-waving: it is known that every interval of real numbers contains both rational and irrational numbers. This implies that we can find tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ as above and with norms smaller than $\delta$, for each $\delta > 0$.

Since $f$ is Riemann integrable, it satisfies the conditions in Definition 2 for every $\varepsilon > 0$ and so in particular for $\varepsilon = 1/2$. This means that there exists $\delta_{1/2} > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[0, 1]$ as above with $\|\dot{\mathcal{P}}\| < \delta_{1/2}$ and $\dot{\mathcal{Q}}$ is any tagged partition of $[0, 1]$ as above with $\|\dot{\mathcal{Q}}\| < \delta_{1/2}$, then

$$|1 - L| = |S(f, \dot{\mathcal{P}}) - L| < \frac{1}{2} \quad \text{and} \quad |0 - L| = |S(f, \dot{\mathcal{Q}}) - L| < \frac{1}{2}. \tag{2}$$

Summing up the two inequalities in (2), we have that

$$|1 - L| + |L| < \frac{1}{2} + \frac{1}{2} = 1.$$

Moreover, the triangle inequality tells us that $|x + y| \leq |x| + |y|$, for each real numbers $x$ and $y$. Therefore, using this with $x = 1 - L$ and $y = L$, we obtain

$$1 = |(1 - L) + L| \leq |1 - L| + |L| < 1,$$

which is clearly a nonsense. This means that our assumption "$f$ is Riemann integrable" was wrong and so $f$ is not Riemann integrable.

We have somehow convinced ourselves that working with the definition of Riemann integrability is not easy, the main reason being that we have to "guess" the "limit" $L$. There are then several difficulties in working with tagged partitions. Indeed, for each partition, there are infinitely many choices of tags and there are infinitely many partitions having a norm less than a specified amount. Luckily, it is possible to establish powerful tools for proving integrability. For example, it can be showed that a given function is Riemann integrable if and only if it can be "squeezed" between two Riemann integrable functions with sufficient accuracy. We will not precisely formulate this criterion and we just remark that it can be used to prove the following important results:

**Theorem 6.** *If $f\colon [a, b] \to \mathbb{R}$ is a monotone function on $[a, b]$, then it is Riemann integrable on $[a, b]$.*

**Theorem 7.** *If $f\colon [a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$, then it is Riemann integrable on $[a,b]$.*

**Theorem 8 (Additivity Theorem).** *If $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function on $[a,b]$ and $c \in (a,b)$, then its restrictions to $[a,c]$ and $[c,b]$ are both Riemann integrable and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

**Corollary 9.** *If $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function on $[a,b]$ and $[c,d] \subseteq [a,b]$, then the restriction of $f$ to $[c,d]$ is Riemann integrable on $[c,d]$.*

Until now, we have considered the Riemann integral over an interval $[a,b]$ where $a < b$. It is in fact convenient to have the integral defined more generally:

**Definition 5.** If $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function on $[a,b]$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we let

$$\int_\beta^\alpha f = -\int_\alpha^\beta f \quad \text{and} \quad \int_\alpha^\alpha f = 0.$$

## 2 Fundamental Theorem of Calculus

The derivative and the integral have been independently defined. The notion of derivative was motivated by the problem of finding tangent lines and it is defined in terms of limits of difference quotients. The notion of integral was motivated by the problem of measuring areas under graphs of functions and it is defined in terms of "limits" of finite sums (Riemann sums). The Fundamental Theorem of Calculus reveals the inverse relationship between these two notions.

Given a continuous function $f\colon [a,b] \to \mathbb{R}$, Theorem 7 and Corollary 9 guarantee us that $f$ is Riemann integrable on $[a,x]$ for each $x \in [a,b]$. Therefore, we can define a new function $F\colon [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

**Theorem 10 (Fundamental Theorem of Calculus, Part 1).** *If $f\colon [a,b] \to \mathbb{R}$ is a continuous function and $F\colon [a,b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then $F$ is differentiable and $F'(x) = f(x)$ for each $x \in [a,b]$.*

Theorem 10 implies that in order to find $F$ we have to look among the functions whose derivative is $f$.

**Definition 6.** An **antiderivative** of a function $f\colon [a,b] \to \mathbb{R}$ is a function $G\colon [a,b] \to \mathbb{R}$ such that $G'(x) = f(x)$ for each $x \in [a,b]$.

It is sometimes easy to guess an antiderivative of a function $f$. If we are able to do so, then we are able to determine the integral of $f$:

**Theorem 11 (Fundamental Theorem of Calculus, Part 2).** *If $f\colon [a,b] \to \mathbb{R}$ is a continuous function and $G\colon [a,b] \to \mathbb{R}$ an antiderivative of $f$, then*

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a).$$

*Proof.* By Theorem 10, the function $F$ given by $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f$. Since $G$ is another antiderivative of $f$, we have that $G'(x) = f(x) = F'(x)$ for each $x \in [a,b]$ and so

$$(F - G)'(x) = F'(x) - G'(x) = 0$$

for each $x \in [a,b]$. This implies that $F - G$ is a constant function on $[a,b]$, i.e. there exists a constant $C \in \mathbb{R}$ such that $F(x) - G(x) = C$ for each $x \in [a,b]$. Therefore, since $F(a) = \int_a^a f(t)\,\mathrm{d}t = 0$, we have that $C = F(a) - G(a) = -G(a)$. But then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) = G(b) + C = G(b) - G(a),$$

as desired. $\square$

**Example 12.** An antiderivative of $f(x) = e^x$ is $G(x) = e^x$. Therefore,

$$\int_0^a e^x \, \mathrm{d}x = G(a) - G(0) = e^a - 1.$$

**Example 13.** An antiderivative of $f(x) = mx$ is $G(x) = mx^2/2$. Therefore,

$$\int_a^b mx \, \mathrm{d}x = G(b) - G(a) = m\frac{b^2 - a^2}{2}.$$

Geometrically, this is the area of a trapezoid with height $b - a$ and bases $ma$ and $mb$.

**Example 14.** Compute $I = \int_0^\pi \sin x \, \mathrm{d}x$. Since an antiderivative of $f(x) = \sin x$ is $G(x) = -\cos x$, we have that $I = (-\cos \pi) - (-\cos 0) = 2$.

**Example 15.** Compute $I = \int_0^1 \frac{1}{1+x^2} \, \mathrm{d}x$. Since an antiderivative of $f(x) = \frac{1}{1+x^2}$ is $G(x) = \arctan x$, we have that $I = \arctan 1 - \arctan 0 = \pi/4$.

**Example 16.** Compute $I = \int_0^\pi \sin x + 3x^4 \, \mathrm{d}x$. By Theorem 3, we have that

$$I = \int_0^\pi \sin x \, \mathrm{d}x + \int_0^\pi 3x^4 \, \mathrm{d}x = 2 + 3\pi^5/5.$$

**Example 17.** Compute $I = \int_0^{3\pi} |\sin x| \, \mathrm{d}x$. By the Additivity Theorem and the fact that $f(x) = \sin x$ is positive on $[0, \pi]$ and $[2\pi, 3\pi]$ and negative on $[\pi, 2\pi]$, we have

$$I = \int_0^\pi \sin x \, \mathrm{d}x + \int_\pi^{2\pi} -\sin x \, \mathrm{d}x + \int_{2\pi}^{3\pi} \sin x \, \mathrm{d}x = 2 + 2 + 2 = 6.$$

## 2.1 Areas Between Curves

In this section, we are interested in determining the area beteween two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. If $f$ and $g$ are both non-negative on $[a, b]$ and $g$ lies below $f$ on that same interval (i.e., $g(x) \leq f(x)$ for each $x \in [a, b]$), then we naturally assign the value

$$\int_a^b f(x) \, \mathrm{d}x - \int_a^b g(x) \, \mathrm{d}x = \int_a^b f(x) - g(x) \, \mathrm{d}x$$

to this area. Note that this is true even if $f$ and $g$ are sometimes negative. Indeed, we can always find a real number $K$ such that $f + K$ and $g + K$ are both non-negative on $[a, b]$. But then the region between $y = f(x)$ and $y = g(x)$ has the same area as the region between $y = f(x) + K$ and $y = g(x) + K$, and this common value is given by

$$\int_a^b f(x) + K \, \mathrm{d}x - \int_a^b g(x) + K \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x - \int_a^b g(x) \, \mathrm{d}x = \int_a^b f(x) - g(x) \, \mathrm{d}x.$$

The following criterion is useful in order to establish that a given curve $y = f(x)$ lies below a curve $y = g(x)$ on a given interval:

**Lemma 18 (Comparison Criterion).** *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be two differentiable functions on $[a, b]$. If $f(a) \leq g(a)$ and $f'(x) \leq g'(x)$ for each $x \in [a, b]$, then $f(x) \leq g(x)$ for each $x \in [a, b]$.*

In words, the Comparison Criterion tells us that if the value of $f$ is at most the value of $g$ at the left endpoint $a$ and the function $g$ increases faster than $f$, then $f$ is smaller than $g$ throughout the interval $[a, b]$.

**Exercise 19.** Show why Lemma 18 is true.

**Example 20.** Find the area between the curves $y = x$ and $y = \sin x$ for $0 \le x \le \pi/4$.

We first observe that $\sin x \le x$ for each $x \in [0, \pi/4]$. This follows from the Comparison Criterion and the fact that $0 \le 0$ and $\cos x \le 1$ for each $x \in [0, \pi/4]$. Therefore, the desired area is the difference of the areas under the bigger curve and the smaller curve, i.e.

$$\int_0^{\pi/4} x - \sin x \, \mathrm{d}x = \int_0^{\pi/4} x \, \mathrm{d}x - \int_0^{\pi/4} \sin x \, \mathrm{d}x = \pi^2/32 + \sqrt{2}/2 - 1.$$

**Example 21.** Find the area between the curves $y = x^2$ and $y = x^3$ for $0 \le x \le 1$.

Note that if $0 \le x \le 1$, then $0 \le x^3 \le x^2$ and so $y = x^3$ lies below $y = x^2$. This means that the desired area is

$$\int_0^1 x^2 \, \mathrm{d}x - \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{3} - \frac{1}{4}.$$

**Example 22.** Find the area between the curves $y = e^x$ and $y = x$ for $0 \le x \le 1$.

Note that $0 \le e^0$ and $1 \le e^x$ for each $x \in [0, 1]$. Therefore, the Comparison Criterion tells us that the curve $y = x$ lies below $y = e^x$ on $[0, 1]$ and so the desired area is

$$\int_0^1 e^x \, \mathrm{d}x - \int_0^1 x \, \mathrm{d}x = e - 1 - \frac{1}{2}.$$

**Example 23.** Find the area between the curves $y = x^3 - x$ and $y = x^2$.

Let us determine the points of intersection of the two curves. We need to solve $x^3 - x = x^2$ or, equivalently, $x^3 - x^2 - x = 0$. This gives the solutions $x = 0$, $x = (1 + \sqrt{5})/2$ and $x = (1 - \sqrt{5})/2$.

It is easy to see that the curve $y = x^2$ lies below $y = x^3 - x$ on $[(1 - \sqrt{5})/2, 0]$, whereas $y = x^3 - x$ lies below $y = x^2$ on $[0, (1 + \sqrt{5})/2]$. Therefore, the desired area is

$$\int_{(1-\sqrt{5})/2}^{0} x^3 - x - x^2 \, \mathrm{d}x + \int_0^{(1+\sqrt{5})/2} x^2 - (x^3 - x) \, \mathrm{d}x.$$

**Example 24.** Find the area between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi/2$.

Do the two curves intersect in $[0, \pi/2]$? It is easily seen that $x = \pi/4$ is a solution of $\sin x = \cos x$. On the other hand, the Comparison Criterion tells us that $y = \sin x$ lies below $y = \cos x$ on $[0, \pi/4]$ and that $y = \cos x$ lies below $y = \sin x$ on $[\pi/4, \pi/2]$. Therefore, the desired area is given by

$$\int_0^{\pi/4} \cos x - \sin x \, \mathrm{d}x + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, \mathrm{d}x = 2\sqrt{2} - 2.$$

**Example 25.** Find the area of the region enclosed by the graphs of $y = 8 - x^2$, $y = 7x$ and $y = 2x$ in the first quadrant. Answer: $31/6$.

**Example 26.** Find the area between the curves $y = x - 1$ and $y^2 = 2x + 6$. Answer: $18$.

# 3 Techniques of Integration

In this section we introduce some techniques allowing to compute antiderivatives in a sort of mechanical way. Thanks to Theorem 11, this will allow us to compute integrals as well. For these purposes, it is useful to first introduce the following notion:

**Definition 7.** Given a continuous function $f\colon [a, b] \to \mathbb{R}$, the **indefinite integral** of $f$, denoted by $\int f(x) \, \mathrm{d}x$ or simply $\int f$, is the set of all antiderivatives of $f$.

Fo example, $\int \cos x$ denotes the set of all functions $f$ such that there exists a real number $c$ such that $f(x) = \sin x + c$ for each $x$. Indeed, we have seen in the proof of Theorem 11 that the antiderivatives of a given function all differ by a constant. It is therefore evocative to adopt the following slight abuse of notation: if $F(x)$ is an antiderivative of $f(x)$, we denote the indefinite integral $\int f(x) \, \mathrm{d}x$ by $F(x) + C$.

**Example 27.** Let $f$ and $g$ be two functions. If $F$ is an antiderivative of $f$ and $G$ is an antiderivative of $g$, then $F+G$ is an antiderivative of $f+g$, as $(F+G)' = F' + G' = f + g$. Therefore, we have the evocative formula

$$\int f + g = \int f + \int g.$$

This equality should be interpreted as meaning that an antiderivative of $f+g$ can be obtained by adding an antiderivative of $f$ to an antiderivative of $g$.

**Exercise 28.** Let $k \in \mathbb{R}$ and $f$ be a function. Show that

$$\int k \cdot f = k \int f.$$

## 3.1 Integration by Parts

The first technique comes from transposing the product formula for the derivative to the integral setting:

**Theorem 29 (Integration by parts).** *Let $f$ and $g$ be two differentiable functions such that $f'$ and $g'$ are continuous. Then*

$$\int fg' = fg - \int f'g$$

*and*

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x.$$

*Proof.* By the formula for the derivative of a product of two functions, we have $(fg)' = f'g + fg'$ or, equivalently, $fg' = (fg)' - f'g$. But then Example 27 and Exercise 28 imply that

$$\int fg' = \int ((fg)' - f'g) = \int (fg)' - \int f'g,$$

and since $fg$ can be chosen as one of the functions denoted by $\int (fg)'$, we obtain the desired equality.

The second equality follows from Theorem 11 and it is left as an exercise. □

Integration by parts transfers the difficulty of computing $\int fg'$ to that of computing $\int f'g$. By properly choosing $f$ and $g$, the last integral may be easier than the original one.

> Integration by parts is useful when the function to be integrated is a product of a function $f$, whose derivative is simpler than $f$, and a function of the form $g'$.

**Example 30.** Compute $\int xe^x$.

$$\int \underset{\substack{\downarrow \\ f}}{x} \cdot \underset{\substack{\downarrow \\ g'}}{e^x} = \underset{\substack{\downarrow \\ f}}{x} \cdot \underset{\substack{\downarrow \\ g}}{e^x} - \int \underset{\substack{\downarrow \\ f'}}{1} \cdot \underset{\substack{\downarrow \\ g}}{e^x} = xe^x - e^x + c.$$

**Example 31.** Compute $\int x \sin x$.

$$\int \underset{\substack{\downarrow \\ f}}{x} \cdot \underset{\substack{\downarrow \\ g'}}{\sin x} = \underset{\substack{\downarrow \\ f}}{x} \cdot \underset{\substack{\downarrow \\ g}}{(-\cos x)} - \int \underset{\substack{\downarrow \\ f'}}{1} \cdot \underset{\substack{\downarrow \\ g}}{(-\cos x)} = -x\cos x + \sin x + c.$$

**Example 32.** Compute $\int x^3 \ln x$.

$$\int x^3 \ln x = \int \underset{\substack{\downarrow \\ g'}}{x^3} \cdot \underset{\substack{\downarrow \\ f}}{\ln x} = \underset{\substack{\downarrow \\ g}}{(x^4/4)} \cdot \underset{\substack{\downarrow \\ f}}{\ln x} - \int \underset{\substack{\downarrow \\ g}}{(x^4/4)} \cdot \underset{\substack{\downarrow \\ f'}}{(1/x)} = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c.$$

**Example 33.** Compute $\int (1+2x^2)e^{x^2}$.

$$\int (1+2x^2)e^{x^2} = \int e^{x^2} + \int 2x^2e^{x^2} \quad \text{and} \quad \int 2x^2 e^{x^2} = \int \underset{\underset{f}{\downarrow}}{x} \cdot \underset{\underset{g'}{\downarrow}}{2xe^{x^2}} = \underset{\underset{f}{\downarrow}}{x} \cdot \underset{\underset{g}{\downarrow}}{e^{x^2}} - \int \underset{\underset{f'}{\downarrow}}{1} \cdot \underset{\underset{g}{\downarrow}}{e^{x^2}}.$$

Therefore,

$$\int (1+2x^2)e^{x^2} = xe^{x^2} + c.$$

One useful trick is to consider the function $g'$ to be the factor $1$.

**Example 34.** Compute $\int \ln x$.

$$\int \ln x = \int \underset{\underset{g'}{\downarrow}}{1} \cdot \underset{\underset{f}{\downarrow}}{\ln x} = \underset{\underset{g}{\downarrow}}{x} \cdot \underset{\underset{f}{\downarrow}}{\ln x} - \int \underset{\underset{g}{\downarrow}}{x} \cdot \underset{\underset{f'}{\downarrow}}{(1/x)} = x\ln x - x + c.$$

**Example 35.** Compute $\int \ln^2 x$.

$$\int \ln^2 x = \int \underset{\underset{g'}{\downarrow}}{1} \cdot \underset{\underset{f}{\downarrow}}{\ln^2 x} = \underset{\underset{g}{\downarrow}}{x} \cdot \underset{\underset{f}{\downarrow}}{\ln^2 x} - \int \underset{\underset{g}{\downarrow}}{x} \cdot \underset{\underset{f'}{\downarrow}}{(1/x)2\ln x} = x\ln^2 x - 2\int \ln x = x\ln^2 x - 2(x\ln x - x) + c.$$

**Example 36.** Compute $\int \arctan x$.

$$\int \arctan = \int \underset{\underset{g'}{\downarrow}}{1} \cdot \underset{\underset{f}{\downarrow}}{\arctan x} = \underset{\underset{g}{\downarrow}}{x} \cdot \underset{\underset{f}{\downarrow}}{\arctan x} - \int \underset{\underset{g}{\downarrow}}{x} \cdot \underset{\underset{f'}{\downarrow}}{(1/1+x^2)} = x\arctan x - \frac{1}{2}\ln(1+x^2) + c.$$

Another trick is to use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve for $\int h$.

**Example 37.** Compute $\int (1/x)\ln x$.

$$\int \underset{\underset{g'}{\downarrow}}{(1/x)} \cdot \underset{\underset{f}{\downarrow}}{\ln x} = \underset{\underset{f}{\downarrow}}{\ln x} \cdot \underset{\underset{g}{\downarrow}}{\ln x} - \int \underset{\underset{f'}{\downarrow}}{(1/x)} \cdot \underset{\underset{g}{\downarrow}}{\ln x},$$

which implies that $2\int (1/x)\cdot \ln x = \ln^2 x$ and so

$$\int (1/x)\cdot \ln x = \frac{1}{2}\ln^2 x + c.$$

**Example 38.** Compute $\int e^x \sin x$.

$$\int \underset{\underset{f}{\downarrow}}{e^x} \cdot \underset{\underset{g'}{\downarrow}}{\sin x} = \underset{\underset{f}{\downarrow}}{e^x} \cdot \underset{\underset{g}{\downarrow}}{(-\cos x)} - \int \underset{\underset{f'}{\downarrow}}{e^x} \cdot \underset{\underset{g}{\downarrow}}{(-\cos x)} = -e^x\cos x + \int e^x \cos x.$$

Now we use integration by parts again on $\int e^x \cos x$ and obtain

$$\int \underset{\underset{f}{\downarrow}}{e^x} \cdot \underset{\underset{g'}{\downarrow}}{\cos x} = \underset{\underset{f}{\downarrow}}{e^x} \cdot \underset{\underset{g}{\downarrow}}{\sin x} - \int \underset{\underset{f'}{\downarrow}}{e^x} \cdot \underset{\underset{g}{\downarrow}}{\sin x}.$$

Therefore,

$$\int e^x \sin x = -e^x \cos x + e^x \sin x - \int e^x \sin x$$

and finally

$$\int e^x \sin x = \frac{e^x(\sin x - \cos x)}{2} + c.$$

**Example 39.** Compute $\int \sin^2 x$.

$$\int \sin^2 x = \int \underset{f}{\underset{\downarrow}{\sin x}} \cdot \underset{g'}{\underset{\downarrow}{\sin x}} = \underset{f}{\underset{\downarrow}{\sin x}} \cdot \underset{g}{\underset{\downarrow}{(-\cos x)}} - \int \underset{f'}{\underset{\downarrow}{\cos x}} \cdot \underset{g}{\underset{\downarrow}{(-\cos x)}} = -\sin x \cos x + \int \cos^2 x.$$

Since $\sin^2 x + \cos^2 x = 1$, we have

$$\int \sin^2 x = -\sin x \cos x + \int (1 - \sin^2 x) = -\sin x \cos x + x - \int \sin^2 x$$

and so

$$\int \sin^2 x = \frac{x - \sin x \cos x}{2} + c.$$

Arguably the fastest way of computing $\int \sin^2 x$ is by recalling that

$$\boxed{\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \quad \text{and} \quad \sin 2x = 2 \sin x \cos x.}$$

Indeed,

$$\int \sin^2 x = \frac{1}{2} \int (1 - \cos 2x) = \frac{1}{2}x - \frac{1}{4} \int 2 \cos 2x = \frac{1}{2}x - \frac{1}{4} \sin 2x = \frac{x - \sin x \cos x}{2} + c.$$

Let's see yet another way of computing $\int \sin^2 x$, this time by a lengthy repeated application of integration by parts:

$$\int \sin^2 x = \int \underset{g'}{\underset{\downarrow}{1}} \cdot \underset{f}{\underset{\downarrow}{\sin^2 x}} = \underset{g}{\underset{\downarrow}{x}} \cdot \underset{f}{\underset{\downarrow}{\sin^2 x}} - \int \underset{f'}{\underset{\downarrow}{2 \sin x \cos x}} \cdot \underset{g}{\underset{\downarrow}{x}} = x \sin^2 x - \int x \sin 2x.$$

We now need to compute $\int x \sin 2x$:

$$\int x \sin 2x = \frac{1}{2} \int 2 \cdot x \sin 2x = \frac{1}{2} \int \underset{f}{\underset{\downarrow}{x}} \cdot \underset{g'}{\underset{\downarrow}{2 \sin 2x}} = \frac{1}{2}\left( \underset{f}{\underset{\downarrow}{x}} \cdot \underset{g}{\underset{\downarrow}{(-\cos 2x)}} - \int \underset{f'}{\underset{\downarrow}{1}} \cdot \underset{g}{\underset{\downarrow}{(-\cos 2x)}} \right) = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + c.$$

Combining everything, we obtain

$$\int \sin^2 x = x \sin^2 x - \int x \sin 2x = x \sin^2 x + \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x + c$$

and using the trigonometric formulas above it is easy to see that this expression simplifies to $\frac{x - \sin x \cos x}{2} + c$.

## 3.2 Substitution Rule

The second technique comes from transposing the Chain Rule (the rule for computing the derivative of a composition of two functions) to the integral setting:

**Theorem 40 (Substitution Rule).** *Let $f$ be a continuous function and $g$ be a differentiable function such that $g'$ is continuous. If $F$ is an antiderivative of $f$, then*

$$\int f(g(x)) \cdot g'(x)\ \mathrm{d}x = F(g(x)) + c.$$

*Moreover,*

$$\int_a^b f(g(x)) \cdot g'(x)\ \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u.$$

*Proof.* By the Chain Rule, we have that the derivative of $F(g(x))$ is $F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$ and so $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ and we have

$$\int f(g(x)) \cdot g'(x)\ \mathrm{d}x = F(g(x)) + c.$$

Let us now prove the formula about definite integrals. Since $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$, Theorem 11 implies that

$$\int_a^b f(g(x)) \cdot g'(x)\ \mathrm{d}x = F(g(b)) - F(g(a)).$$

Moreover, since $F$ is an antiderivative of $f$, again by Theorem 11, we have that

$$\int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u = F(g(b)) - F(g(a))$$

and so the desired equality holds. □

The simplest use of the Substitution Rule is by recognizing that a given function is of the form $f(g(x)) \cdot g'(x)$.

**Example 41.** Compute $\int_a^b \sin^5 x \cos x\ \mathrm{d}x$.

Since $\sin^5 x \cos x$ can be written as $f(g(x)) \cdot g'(x)$ by taking $g(x) = \sin x$ and $f(x) = x^5$, we have

$$\int_a^b \sin^5 x \cos x\ \mathrm{d}x = \int_a^b f(g(x)) \cdot g'(x)\ \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u = \int_{\sin a}^{\sin b} u^5\ \mathrm{d}u = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6}.$$

**Example 42.** Compute $\int_a^b \tan x\ \mathrm{d}x$.

Since $\tan x = \frac{\sin x}{\cos x}$, we have that $-\tan x = \frac{-\sin x}{\cos x}$ can be written as $f(g(x)) \cdot g'(x)$ by taking $g(x) = \cos x$ and $f(x) = 1/x$. Therefore,

$$\int_a^b -\tan x\ \mathrm{d}x = \int_a^b f(g(x)) \cdot g'(x)\ \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u = \int_{\cos a}^{\cos b} \frac{1}{u}\ \mathrm{d}u = \ln(\cos b) - \ln(\cos a)$$

and so $\int_a^b \tan x\ \mathrm{d}x = \ln(\cos a) - \ln(\cos b)$.

Note that the use of the Substitution Rule

$$\int_a^b f(g(x)) \cdot g'(x)\ \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u$$

can be shortened by adopting the following simple mechanical process: To go from the left side to the right side, we substitute $u$ for $g(x)$ and $\mathrm{d}u$ for $g'(x)\ \mathrm{d}x$ and change the interval of integration. This can be further abbreviated by simply saying: Let $u = g(x)$ and $\mathrm{d}u = g'(x)\ \mathrm{d}x$.

**Remark 8.** We have attributed no meanings to the symbols $\mathrm{d}u$ and $\mathrm{d}x$ by themselves. They are used as formal devices to help us perform this operation in a mechanical way.

For example, in $\int_a^b \sin^5 x \cos x \, \mathrm{d}x$, we let $u = \sin x$ and $\mathrm{d}u = \cos x \, \mathrm{d}x$ and obtain

$$\int_a^b \sin^5 x \cos x \, \mathrm{d}x = \int_{\sin a}^{\sin b} u^5 \, \mathrm{d}u = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6}.$$

Similarly, in $\int_a^b \tan x \, \mathrm{d}x$, we let $u = \cos x$ and $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and obtain

$$\int_a^b \tan x \, \mathrm{d}x = \int_a^b \frac{\sin x}{\cos x} \, \mathrm{d}x = -\int_{\cos a}^{\cos b} \frac{1}{u} \, \mathrm{d}u = \ln(\cos a) - \ln(\cos b).$$

In the case of indefinite integrals the process is similar and can be summarized as follows:

1. Let $u = g(x)$ and $\mathrm{d}u = g'(x) \, \mathrm{d}x$. After this manipulation only the letter $u$ should appear in the new integrand.
2. Find an antiderivative (as an expression involving $u$).
3. Substitute $g(x)$ back for $u$.

For example, in order to find $\int \sin^5 x \cos x \, \mathrm{d}x$ we proceed as follows:

1. Let $u = \sin x$ and $\mathrm{d}u = \cos x \, \mathrm{d}x$. We obtain $\int u^5 \, \mathrm{d}u$.
2. Find an antiderivative: $\int u^5 \, \mathrm{d}u = \frac{u^6}{6} + c$.
3. Substitute $g(x)$ back for $u$: $\int \sin^5 x \cos x \, \mathrm{d}x = \frac{\sin^6 x}{6} + c$.

**Example 43.** Compute $\int \frac{x}{\sqrt{1-4x^2}} \, \mathrm{d}x$.

Note that $\frac{-8x}{\sqrt{1-4x^2}}$ can be written as $f(g(x)) \cdot g'(x)$ by taking $g(x) = 1 - 4x^2$ and $f(u) = u^{-1/2}$. Therefore, we proceed as follows:

1. Let $u = 1 - 4x^2$ and $\mathrm{d}u = -8x \, \mathrm{d}x$. The new integral becomes $-\frac{1}{8} \int u^{-1/2} \, \mathrm{d}u$.
2. Find an antiderivative: $-\frac{1}{8} \int u^{-1/2} \, \mathrm{d}u = -\frac{1}{8} \cdot 2 \cdot u^{1/2} + c$.
3. Substitute $g(x)$ back for $u$: $\int \frac{x}{\sqrt{1-4x^2}} \, \mathrm{d}x = -\frac{1}{4}\sqrt{1-4x^2} + c$.

**Example 44.** Compute $\int (x^6 + x^3) \sqrt[3]{x^3 + 2} \, \mathrm{d}x$.

The trick is to bring a factor of $x$ inside the cube root:

$$\int (x^6 + x^3) \sqrt[3]{x^3 + 2} \, \mathrm{d}x = \int (x^5 + x^2) \sqrt[3]{x^6 + 2x^3} \, \mathrm{d}x.$$

Now we let $u = x^6 + 2x^3$ and $\mathrm{d}u = 6x^5 + 6x^2 \, \mathrm{d}x$. Therefore, we are left with computing

$$\frac{1}{6} \int u^{1/3} \, \mathrm{d}u = \frac{1}{6} \cdot \frac{3}{4} \cdot u^{4/3} + c = \frac{1}{8} \cdot u^{4/3} + c.$$

Substituting $g(x)$ back, we obtain

$$\int (x^6 + x^3) \sqrt[3]{x^3 + 2} \, \mathrm{d}x = \frac{1}{8} \cdot (x^6 + 2x^3)^{4/3} + c.$$

Success in using the Substitution Rule depends on the ability of determining which part of the integrand should be replaced by the symbol $u$. In fact, the most interesting uses of the Substitution Rule occur when the factor $g'(x)$ does not appear. There are essentially two scenarios that could arise:

Replace a complicated expression by a simpler one.

**Example 45.** Compute $\int \frac{1+e^x}{1-e^x}\,\mathrm{d}x$.
It seems natural to let $u = e^x$ and $\mathrm{d}u = e^x\,\mathrm{d}x$. Note that

$$\int \frac{1+e^x}{1-e^x}\,\mathrm{d}x = \int \frac{1+e^x}{1-e^x} \cdot \frac{1}{e^x} \cdot e^x\,\mathrm{d}x$$

and so we are left with computing

$$\int \frac{1+u}{1-u} \cdot \frac{1}{u}\,\mathrm{d}u = \int \frac{2}{1-u} + \frac{1}{u}\,\mathrm{d}u = 2\int \frac{1}{1-u}\,\mathrm{d}u + \int \frac{1}{u}\,\mathrm{d}u = -2\ln(1-u) + \ln u.$$

The trick used in the first equality above allows to find antiderivatives of rational functions. We will treat this topic more systematically in the following sections. Finally, substituting $g(x)$ back, we obtain

$$\int \frac{1+e^x}{1-e^x}\,\mathrm{d}x = -2\ln(1-e^x) - \ln e^x = -2\ln(1-e^x) - x.$$

Let us see an alternative way of computing the indefinite integral above. Instead of expressing $u$ in terms of $x$ and $\mathrm{d}u$ in terms of $\mathrm{d}x$, we express $x$ in terms of $u$ and $\mathrm{d}x$ in terms of $\mathrm{d}u$. In this case, we let again $u = e^x$ and obtain

$$x = \ln u \quad \text{and} \quad \mathrm{d}x = \frac{1}{u}\,\mathrm{d}u.$$

Note that $\int \frac{1+e^x}{1-e^x}\,\mathrm{d}x$ immediately becomes $\int \frac{1+u}{1-u} \cdot \frac{1}{u}\,\mathrm{d}u$. More generally, if $g$ is a bijective function, then we can apply the substitution

$$x = g(t) \quad \text{and} \quad \mathrm{d}x = g'(t)\mathrm{d}t.$$

**Example 46.** Compute $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,\mathrm{d}x$.
We let $u = \sqrt{e^x+1}$. Therefore, $u^2 = e^x + 1$ and so

$$x = \ln(u^2-1) \quad \text{and} \quad \mathrm{d}x = \frac{2u}{u^2-1}\,\mathrm{d}u.$$

The integral becomes

$$\int \frac{(u^2-1)^2}{u} \cdot \frac{2u}{u^2-1}\,\mathrm{d}u = 2\int u^2 - 1\,\mathrm{d}u = 2\frac{u^3}{3} - 2u + c.$$

Substituting $g(x)$ back, we obtain

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,\mathrm{d}x = \frac{2}{3}(e^x+1)^{3/2} - 2(e^x+1)^{1/2} + c.$$

**Example 47.** Compute $\int x\sqrt{1-x}\,\mathrm{d}x$.
We let $u = 1 - x$ and $\mathrm{d}u = -1\,\mathrm{d}x$. The integral becomes

$$\int -(1-u)\sqrt{u}\,\mathrm{d}u = \int -\sqrt{u}\,\mathrm{d}u + \int u\sqrt{u}\,\mathrm{d}u = -\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c.$$

Substituting $g(x)$ back, we obtain

$$\int x\sqrt{1-x}\,\mathrm{d}x\,\mathrm{d}x = -\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + c.$$

Use a trigonometric substitution.

**Example 48.** Compute $\int \sqrt{1-x^2}\ \mathrm{d}x$.

We let $x = \sin u$ and $\mathrm{d}x = \cos u\ \mathrm{d}u$. Clearly, $u = \arcsin x$. The integral becomes

$$\int \sqrt{1-\sin^2 u} \cos u\ \mathrm{d}u = \int \cos^2 u\ \mathrm{d}u.$$

Recalling that $\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$, we have $\cos^2 u = \frac{\cos 2u + 1}{2}$ and so

$$\int \cos^2 u\ \mathrm{d}u = \int \frac{\cos 2u + 1}{2} = \frac{u}{2} + \frac{\sin 2u}{4} + c.$$

Substituting back, we have

$$\int \sqrt{1-x^2}\ \mathrm{d}x = \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + c.$$

We will see more examples of trigonometric substitutions in the next sections.

## 3.3 Trigonometric Integrals

Read section 7.2 of the book.